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Chaotic itinerancy and power-law residence time distribution in stochastic dynamical systems

Jun Namikawa*
School of Knowledge Science, Japan Advanced Institute of Science and Technology, Tatsunokuchi, Ishikawa 923-1292, Japan
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Chaotic itinerant motion among varieties of ordered states is described by a stochastic model based on the mechanism of chaotic itinerancy. The model consists of a random walk on a half-line and a Markov chain with a transition probability matrix. The stability of attractor ruin in the model is investigated by analyzing the residence time distribution of orbits at attractor ruins. It is shown that the residence time distribution averaged over all attractor ruins can be described by the superposition of (truncated) power-law distributions if the basin of attraction for each attractor ruin has a zero measure. This result is confirmed by simulation of models exhibiting chaotic itinerancy. Chaotic itinerancy is also shown to be absent in coupled Milnor attractor systems if the transition probability among attractor ruins can be represented as a Markov chain.

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The concept of chaotic itinerancy (CI) has been proposed as a universal class of dynamics in multiattractor systems to explain phenomena such as chaotic itinerant motion among varieties of ordered states [1–3]. CI describes the attraction of dynamical orbits to an ordered motion state, which is maintained for a certain length of time but eventually degenerates into high-dimensional chaotic motion by separation from the ordered state. Repetition of this process results in successive itineration over ordered motion. The ordered motion states in this sequence are referred to as "attractor ruin." Demonstrating the mathematical properties of CI has become an important problem for a variety of systems in many scientific disciplines, including semiconductor physics, chemistry, neuroscience, and laser physics [4].

To mathematically characterize CI, some researchers have suggested that attractor ruin should be represented as Milnor attractors [4–7], that is, a minimal invariant set with a positive measure as its basin of attraction [8]. As this definition does not exclude the possibility of orbits leaving from any neighborhood of the attractor, attractor ruin can be validly described in terms of Milnor attractors. Several models in which the existence of Milnor attractors with CI have been reported [5–7]. However, it remains unclear as to whether Milnor attractors must exist in any system exhibiting CI.

The stability of attractor ruin is an important property for the characterization of CI, and the residence time distribution is regarded as the statistical property of such stability. In this paper, the residence time distribution of orbits at attractor ruin are investigated, and the possibility of CI in a dynamical system containing Milnor attractors is discussed. A mechanism of transition among attractor ruins is introduced to describe the distribution, and a model based on that mechanism is proposed.

A globally coupled map (GCM) is employed as an example of a dynamical system displaying CI [2], as described by

$$x_{t+1}(i) = (1-\epsilon)f(x_t(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_t(j)), \tag{1}$$

where $x_t(i)$ is a real number, t is a discrete time step, i is the index for elements ($i=1,2,\ldots,N=$system size) and f is a map on $\mathbb{R}$ as the local element in Eq. (1). When the elements i and j are synchronized, i.e., $x_t(i)\approx x_t(j)$, the elements are part of a single cluster. Each element belongs to a cluster, including clusters with only one element. The state of the GCM is characterized by the partition of elements into clusters, and the state of the partition is called the clustering condition. CI in the GCM is observed as chaotic changes of the clustering conditions [5,9], that is, a clustering condition implies attractor ruin. If all the elements of a cluster in a clustering condition are fully synchronized, i.e., $x_t(i)=x_t(j)$ for any elements in one cluster, the clustering condition is an invariant subspace. Therefore, CI in a GCM is considered to represent the phenomenon in which orbits approach an invariant subspace after lingering in the neighborhood of another invariant subspace for some time. As an indicator of the stability of clustering conditions, the local splitting exponent $\lambda_{\mathrm{spl}}^{T}(i,n)$ [10] is introduced, as defined by

$$\lambda_{\mathrm{spl}}^{T}(i,n) = \frac{1}{T}\sum_{m=n}^{n+T} \ln\left|(1-\epsilon)Df(x_m(i))\right|. \tag{2}$$

The local splitting exponent represents the local expansion rate between an element i and an adjacent element, and thus can be considered to represent the local expansion rate between two elements in the cluster to which the element i belongs. The element i is contained in a cluster with more than one element if $\lambda_{\mathrm{spl}}^{T}(i,n)$ is negative, and the element does not synchronize with any other elements if $\lambda_{\mathrm{spl}}^{T}(i,n)$ is positive. Hence, switching of the sign of $\lambda_{\mathrm{spl}}^{T}(i,n)$ indicates a change in the clustering condition. Moreover, if an orbit approaches sufficiently close to an invariant subspace corresponding to a clustering condition, the local splitting exponent can describe the distance between the orbit and the invariant subspace using a logarithmic scale.

*Electronic address: jnamika@jaist.ac.jp

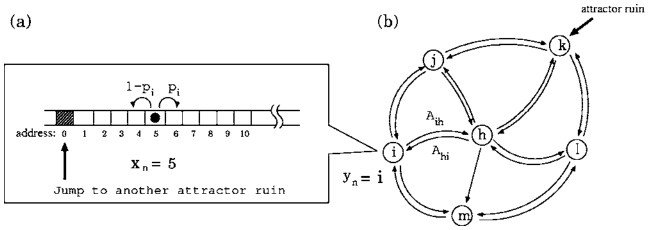

FIG. 1. Conceptual illustration of an itinerant dynamics model. (a) Schematic of the space of x_n, where the stability of an attractor ruin is represented by a random walk on a half-line, and p_i is a probability that a point moves to the right on the half-line, i.e., $x_{n+1}=x_n+1$. If the point arrives at the leftmost address on the half-line, transition among attractor ruins is triggered. (b) Schematic of the space of y_n. Transition among attractor ruins is described by a Markov chain with a transition probability matrix A, where $A_{i,j}$ is the probability of moving from attractor ruin i to j, i.e., $y_n=i$ and $y_{n+1}=j$.

In general, if the system is unstable in the direction normal to an invariant subspace, an orbit leaves an attractor ruin containing the invariant subspace. The mechanism that destabilizes an invariant subspace is referred to as a blowout bifurcation, and is known to cause bursts in on-off intermittency [11].

In summary, the mechanism of CI can be described as follows.

- An orbit leaves an attractor ruin when the distance between the orbit and an invariant subspace in the attractor ruin is greater than a certain value.
- If nonlinearity (i.e., a local expansion rate) is weak in the direction normal to the invariant subspace, the distance between the orbit and the invariant subspace decreases. Otherwise, the distance increases.

Based on this mechanism mentioned, a prototype model in which CI occurs is introduced. Consider the distance between an orbit and the nearest invariant subspace. Although in the GCM the number of values for nonlinearity (i.e., local splitting exponents) is equal to the system size, only one variable is used in the present system to represent nonlinearity, having a stochastic value of 1 or -1 decided by a probability associated with the nearest attractor ruin to the orbit. A probability governing the transition among attractor ruins is also introduced. In the definition of this transition probability, it is assumed that the influence of a past attractor ruin upon the current potential transition decays rapidly, such that the transition depends on only a finite number of past attractor ruins. This assumption affords the simplest case of chaotic itinerant dynamics on attractor ruins in CI.

A stochastic model satisfying the characteristics above is as follows. Let $n\in\mathbb{N}$, $x_n\in\mathbb{N}\cup\{0\}$, and $y_n=\{1,\ldots,M\}$. A series of positive integers $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1}=\begin{cases}0 & x_n=0 \quad \text{and} \quad \epsilon_n=-1,\\ x_n+\epsilon_n & \text{otherwise},\end{cases} \tag{3}$$

where ϵ_n is a stochastic variable with a value of 1 with probability p_{y_n} or -1 with $1-p_{y_n}$. Here, x_n is given by a random walk on a half-line [see Fig. 1(a)]. A series $\{y_n\}_{n=0}^{\infty}$ is also defined, as given by

$$y_{n+1}=\begin{cases}y_n & x_n>0,\\ z_n & \text{otherwise},\end{cases} \tag{4}$$

where z_n is a stochastic variable such as $z_n=k$ with probability A_{y_nk}, being an element of a non-negative square matrix A. The term y_n is determined by a Markov chain with the transition probability matrix A [see Fig. 1(b)], representing an index for attractor ruins, and e^{-x_n} is the distance from an invariant set corresponding to y_n. The variable p_y denotes the intensity of nonlinearity on attractor ruin y, and A_{ij} is the transition probability from attractor ruin i to j.

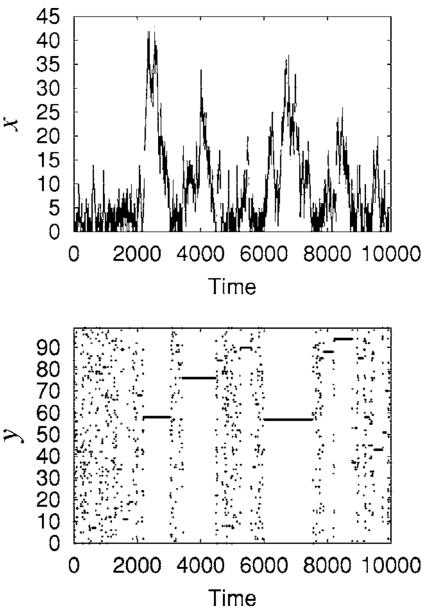

FIG. 2. Time series of x_n and y_n $(M=100, A_{ij}=1/M, p_i=0.45+\mu_i)$.

Figure 2 shows a time series of x_n and y_n with respect to $M=100$, $A_{ij}=1/M$, and $p_i=0.45+\mu_i$, where μ_i is a random number taken from $[-0.05,0.05]$. The figure reveals a region in which y_n is fixed and a region in which y_n changes dynamically. In the former, x_n is large, representing that case where an orbit approaches the invariant set.

The residence time distribution of orbits at an attractor ruin is defined in the present model as follows. The probability $P(i,t)$ for residence time t at an attractor ruin i is given by the probability for $x_{n+t}=0$ if $x_n=0$, $y_n=i$, and $x_{n+k}>0$ for any $k<t$ and $n\in\mathbb{N}$. Since t is regarded as the recurrent time to origin in a one-dimensional random walk [12,13], this probability is given by

$$P(i,t)=\begin{cases}1-p_i & t=1,\\ \dfrac{[p_i(1-p_i)]^n}{n}\dbinom{2n-2}{n-1} & \exists\ n\in\mathbb{N}\ t=2n,\\ 0 & \text{otherwise}.\end{cases} \tag{5}$$

If $t=2n$ and $n\in\mathbb{N}$, $\ln P(i,2n)$ can be denoted by

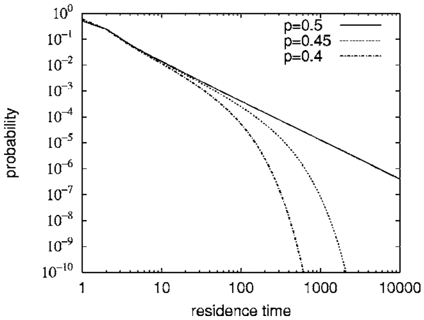

FIG. 3. Residence time distribution at attractor ruin i with probability p_i.

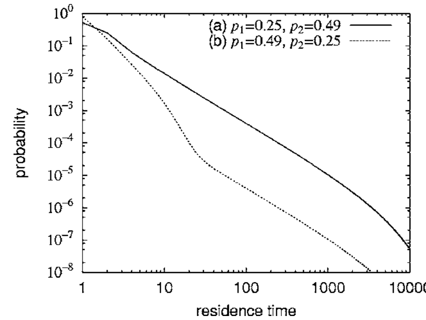

FIG. 4. Residence time distribution averaged over all attractor ruins $(M=2, q_1=0.01, q_2=0.99)$.

$$\ln P(i,2n) \sim n \ln[p_i(1-p_i)] - \frac{1}{2}\ln[\pi n^2(n-1)] + (2n-2)\ln 2 \tag{6}$$

which is given by the approximate expression of Eq. (5). If $p_i=0.5$, then the right-hand side of Eq. (6) takes the form of a constant plus the second term. That is, the residence time at the attractor ruin i is governed by a power-law distribution. If $p_i<0.5$, then n in the first and third terms of Eq. (6) are not cancelled, causing the residence time distribution to be truncated (see Fig. 3). The probability that the residence time at the attractor ruin i is longer than t is given by

$$Q(i,t) = 1 - \sum_{k=1}^{t} P(i,k). \tag{7}$$

It is easy to see that $Q(i,1)=p_i$ and $Q(i,2n+1)=Q(i,2n)$ for any $n \in \mathbb{N}$. Since

$$P(i,2n) = \frac{[p_i(1-p_i)]^n}{2}\left[4\binom{2n-2}{n-1} - \binom{2n}{n}\right], \tag{8}$$

then

$$Q(i,2n) = [1-4p_i(1-p_i)]\sum_{k=1}^{n-1}\frac{[p_i(1-p_i)]^k}{2}\binom{2k}{k} + p_i - 2p_i(1-p_i) + \frac{[p_i(1-p_i)]^n}{2}\binom{2n}{n}. \tag{9}$$

Hence, $\lim_{t\to\infty}Q(i,t)=0$ if $p_i \leqslant 0.5$, and $\lim_{t\to\infty}Q(i,t)>0$ if $p_i>0.5$. Accordingly, while the transition from attractor ruin i surely takes place when $p_i \leqslant 0.5$, transition does not necessarily take place when $p_i>0.5$.

In studying CI, the residence time distribution averaged by all attractor ruins is a more useful property than the residence times of each attractor ruin because the distribution of individual attractor ruins can only be studied when the structure of the invariant sets is clear. However, showing the structure is difficult in high-dimensional dynamical systems. To simplify the present discussion, it is assumed without loss of generality that the transition probability matrix A is irreducible [17]. When the eigenvector associated with a positive maximum eigenvalue (existent by the Perron-Frobenius theorem) is denoted by $\mathbf{r}$, normalization of $\mathbf{r}$ (denoted $\mathbf{q}$) provides the stationary distribution in a Markov chain with the transition probability matrix A.

Suppose that $p_i \leqslant 0.5$ for any $1 \leqslant i \leqslant M$. Since the transition from each attractor ruin always occurs at a probability of unity, orbits itinerate over all attractor ruins. Hence, the probability of residence time t for any attractor ruin can be described by $\sum_{i=1}^{M} q_i P(i,t)$ from the probability $P(i,t)$ and the stationary distribution $\mathbf{q}=\{q_i\}_{i=1}^{M}$. Consequently, the residence time distribution averaged by all attractor ruins is the superposition of (truncated) power-law distributions. As a result, there is a case in which such a distribution does not appear to follow a power law. Figure 4 shows some examples of residence time distributions for two attractor ruins. In case (a), the residence time distribution follows a power law due to the existence of a dominant attractor ruin ($p_2=0.49$ and $q_2=0.99$). However, the residence time distribution in case (b) does not follow a power law. In this case, one attractor ruin rarely attracts orbits, but the orbits that it does attract remain attracted for relatively a long time ($p_1=0.49$ and $q_1=0.01$). The other attractor ruin attracts orbits frequently, but the residence time for each is relatively short ($p_2=0.25$ and $q_2=0.99$). In this case, the residence time distribution has multiple scales.

On the other hand, if $p_i>0.5$, then the probability $Q_i=\lim_{t\to\infty}Q(i,t)$ (providing that no transition will ever occur at attractor ruin i) is a positive value. Then, since the transition probability matrix A is irreducible, the probability R that orbits itinerate over attractor ruins forever is given by

$$R = \lim_{n\to\infty}\left(\sum_{i=1}^{M} q_i(1-Q_i)\right)^n = 0. \tag{10}$$

Therefore, CI can only occur as a transient state in this case.

To confirm this observation that the residence time distribution averaged over all attractor ruins in this prototype model of the CI mechanism can be described by the superposition of (truncated) power-law distributions, a computer simulation for the GCM defined by Eq. (1) was performed. Figure 5 shows the simulated residence time distribution averaged over all clustering conditions in a GCM with $f(x)=1-\alpha x^2$. This result clearly shows that the residence time distribution in the GCM follows a power law as in the proposed model. Figure 5(a) represents the case that orbits itinerate over attractor ruins having under an almost identical

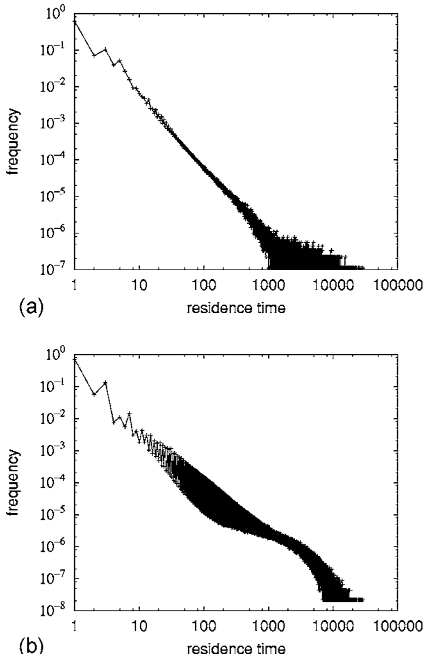

FIG. 5. Residence time distribution averaged over all clustering conditions in a GCM with $N=10$, where elements i and j are synchronized if $|x_t(i)-x_t(j)|<10^{-6}$. (a) $\alpha=1.57$ and $\epsilon=0.3$, (b) $\alpha=1.9$ and $\epsilon=02$.

distribution, and Fig. 5(b) shows the case with different distributions.

As another example of CI, the dynamics of a kicked single rotor under the influence of noise is considered [14]. This dynamics is defined by the following two-dimensional map:

$$x_{n+1}=x_n+y_n+\delta_x \pmod{2\pi},$$

$$y_{n+1}=(1-\nu)y_n+\omega\sin(x_n+y_n)+\delta_y, \tag{11}$$

where x corresponds to the phase, y corresponds to the angular velocity, ν is the damping, and ω is the strength of the forcing. The terms δ_x and δ_y, where $\sqrt{\delta_x^2+\delta_y^2}\leqslant\delta$, are the amplitude of the uniformly and independently distributed noise. The dynamics of Eq. (11) is illustrated in Fig. 6. As seen in Fig. 6(b), the orbit is attracted to certain ordered motion states for a certain period, but is eventually kicked out of the states and enters a chaotic behavior regime. Figure 7 shows the result of a computer simulation using Eq. (11) for the residence time distribution of orbits at attractor ruins. The residence time distribution averaged over all attractor ruins appears to be the superposition of truncated power-law distributions, similar to case (b) in Fig. 4. This simulation and the preceding results imply that two attractor ruins with different residence time distributions exist in the dynamics of Eq. (11). Theoretical results based on the proposed model may be applicable to other models exhibiting CI.

Compare the mechanism of temporal intermittency in low-dimensional dynamical systems with that of CI. Temporal intermittency is a phenomenon in which bursts sometimes appear in the intervals of ordered states, and is seen at some

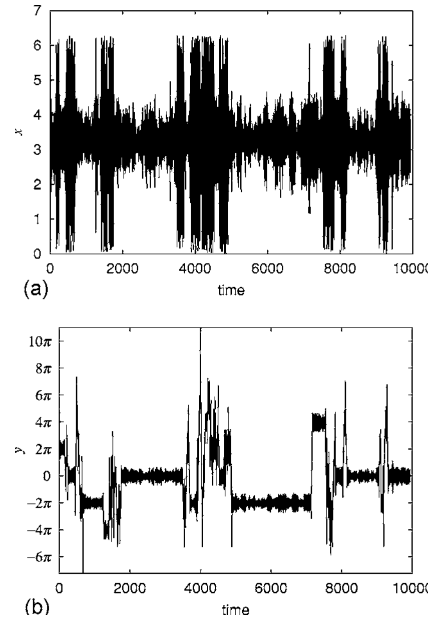

FIG. 6. Example of a time sequence described by Eq. (11) with $\nu=0.02$, $\omega=3.5$, and $\sigma=0.15$. (a) Phase of rotor x, (b) angular velocity y.

points in the parameter space in the neighborhood of bifurcation boundaries [15]. Traditionally, research on temporal intermittency has discussed the occurrence of bursts and recurrence intervals, whereas CI research focuses on chaotic itinerant motions among several ordered states. Phenomenologically, temporal intermittency and CI share several features. For example, ordered states and chaotic states appear in turn. However, as discussed above, the mechanism of CI differs from that of classical temporal intermittency, as seen in the dissimilar residence time distributions [18]. Thus, it is necessary to consider CI as distinct from other temporal intermittencies.

In the present model, for an attractor ruin i with $p_i>0.5$, CI has been shown to occur only as a transient state. Note that $p_i>0.5$ implies that an attractor ruin i is a Milnor attractor, since the basin of attraction for the attractor ruin i has a nonzero measure. Hence, if there exists a Milnor attractor in the proposed model, CI occurs only as a transient state. It has been reported that CI may exist in a coupled Milnor attractor

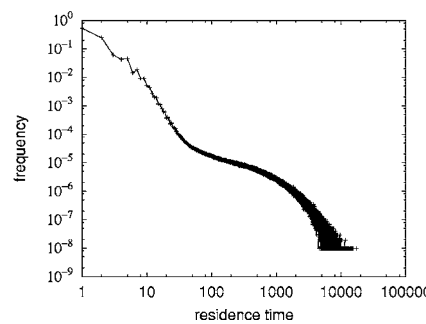

FIG. 7. Residence time distribution averaged over all attractor ruins as given by Eq. (11) with $\nu=0.02$, $\omega=3.5$, and $\sigma=0.15$. Each region corresponding to an attractor ruin is given by $(2n-1)\pi<y\leqslant(2n+1)\pi$ for each $n\in\mathbb{Z}$.

system [7]. From the present results, two possibilities can be considered for CI in such systems: (a) behavior like CI is strictly observed as transient phenomena, or (b) the transition probability among attractor ruins cannot be represented as a Markov chain. In the latter case, it is expected that the system is more complex than considered here, necessitating further analysis of such systems.

This model has good scope for extension. In the present model, the term of nonlinearity has been simplified to either 1 or -1. Although this simplification can be easily removed, it is believed that the same results will be obtained even if the model is extended to the general case. More importantly, the present model does not describe the specific behavior of orbits at each attractor ruin. While the simplified behavior assumed here facilitates investigation of the relationship between the stability and the transition probability of attractor ruins, it prevents discussion of the dynamical behavior at each attractor ruin. One of the extensions to express specific behavior is to prepare a function describing the change of states (dynamics) at any attractor ruin. However, as orbits successively itinerate over attractor ruins, the dynamic change of functions associated with attractor ruins must be considered. Functional shifts provide a framework for describing such dynamical systems [16] by defining a shift space as a set of bi-infinite sequences of some functions on a set of symbols. Functional shifts can be used to represent dynamical systems with dynamic changes of functions. Improving the proposed model by incorporating functional shifts will be a topic of future study.

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[17] A non-negative square matrix A is irreducible iff for any $1 \leqslant i,j \leqslant M$ there exists $n \in \mathbb{N}$, such as $A_{ij}^n > 0$, where A^n is the nth power of A.
[18] Temporal intermittency can be classified into types I, II, and III, where the maximum interval between bursts in type-I intermittency is given by $\epsilon^{-1/2}$ (ϵ positive), while a probability distribution $P(t)$ for the interval between bursts is applicable in type-II and type-III intermittency with either $P(T) \sim e^{-2\epsilon T}$ if $T \gg \epsilon^{-1}$ or $P(T) \sim (4\epsilon T)^{-3/2}$ if $1 \ll T \ll \epsilon^{-1}$.